JOURNAL OF MULTIVARIATE ANALYSIS 3, 226–235 (1973)

The Distribution of the Sphericity Test Criterion

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The exact distribution of Mauchly's sphericity test criterion $W = |\mathbf{S}|/[\operatorname{tr} \mathbf{S}/p]^p$, where $\mathbf{S}$ is the sum of product matrix from a sample of size N taken from a p-variate normal population, is obtained using contour integration and methods similar to those of Nair and Box. Tables of percentage points for $p = 4(1)10$, $\alpha = 0.01$ and 0.05, and various values of N (including small) are given and comparisons made with approximate percentage points using methods of Box, Mauchly, Tukey and Wilks, and Davis.

1. INTRODUCTION

Let $\mathbf{x}: p \times 1$ be distributed $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are both unknown. Let $\mathbf{S}$ be the sum of product matrix of a sample of size N. To test the hypothesis of sphericity, namely, H_0: $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_p$, where $\sigma^2 > 0$ is unknown, against $\boldsymbol{\Sigma} \neq \sigma^2\mathbf{I}_p$, Mauchly [10] obtained the likelihood ratio test criterion for H_0 in the form $W = |\mathbf{S}|/[(\operatorname{tr} \mathbf{S})/p]^p$ and derived its null distribution for $p = 2$. The exact distribution of W in the null case has further been obtained in [3] for $p = 3, 4, 6$ and by Consul [4], Mathai and Rathie [9], and Mathai [8] in the general case, and more recently by John [6] for $p = 3$. The expression in [4] is in terms of G-functions; whereas, the expressions in [8, 9] are in series form. The nonnull distribution of W was first obtained by Pillai and Nagarsenker [14] and later independently by Khatri and Srivastava [7]. No systematic attempt seems to have been made so far to compute the exact percentage points of W.

Received June 21, 1972; revised December 29, 1972.

AMS 1970 subject classifications: Primary 62H10; Secondary 62Q05.

Key words and phrases: Exact distribution; sphericity test criterion; contour integration; factorial series; percentage points; approximations.

* Work of this author was supported in part by the Air Force Aerospace Research Laboratories, Air Force Systems Command, U.S. Air Force, Contract F33615-72-C-1400 and in part by David Ross Grant from Purdue Research Foundation.

† Work of this author was sponsored by the Air Force, Aerospace Research Laboratories, Air Force Systems Command, U.S. Air Force, Contract F33615-72-C-1400. Reproduction in whole or in part is permitted for any purpose of the United States Government.

In this paper methods have been developed similar to the ones used by Box [1, 2] and Nair [12, 13], in order to obtain the exact null distribution of W in series form and to compute exact percentage points of W. The main results are presented in the following section. For the details of the methods, the reader is referred to Nagarsenker and Pillai [11]. Tabulations of 5% and 1% points for $p = 4(1)10$ are given and comparisons made with approximate values using Box's series (Anderson [1, p. 263]), Mauchly [10], Tukey and Wilks [15], and Davis [5].

2. Exact Distribution of the Sphericity Criterion W

For ease in computation, it is desirable to give methods which are particularly suited for extremely small values of N, the sample size, and those which are suited for larger values of N. In parts (a) and (b) of this section, we shall consider two methods which will achieve the former objective while in part (c) we shall give a method which will achieve the latter objective. The first method which we shall now consider makes use of Mellin transform and contour integration.

(a) *Exact Distribution of W by Contour Integration*

It has been shown by Mauchy [10] that the hth moment of the sphericity criterion $W = |\mathbf{S}|/[(\mathrm{tr}\,\mathbf{S})/p]^p$ is given by

$$E(W^h) = p^{ph} \prod_{i=1}^{p} \left[\frac{\Gamma\{\frac{1}{2}(N-i)+h\}}{\Gamma\{\frac{1}{2}(N-i)\}} \frac{\Gamma\{\frac{1}{2}p(N-1)\}}{\Gamma\{\frac{1}{2}p(N-1)+ph\}}\right]. \qquad (2.1)$$

Using Mellin's transform, the density of W is given by

$$f(w) = K(p, n)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{w^{-h-1}p^{ph}\prod_{i=1}^{p}\Gamma\{\frac{1}{2}(N-i)+h\}}{\Gamma\{\frac{1}{2}pn+ph\}}\, dh, \qquad (2.2)$$

where $n = N-1$ and $K(p, n) = \Gamma(\frac{1}{2}pn)/\prod_{i=1}^{p}\Gamma(N-i)/2$. Putting $\frac{1}{2}(N-p)+h = s$ in (2.2), we have

$$f(w) = K(p, n)p^{-(1/2)p(N-p)}w^{(1/2)(N-p)-1}\cdot p(w), \qquad (2.3)$$

where

$$p(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(w/p^p)^{-s}\left[\prod_{i=1}^{p}\Gamma\left(s+\frac{p-i}{2}\right)\Big/\Gamma p\left(s+\frac{p-1}{2}\right)\right] ds, \qquad (2.4)$$

and $c = \frac{1}{2}(N-p)$.

Using a lemma due to Nair [12, p. 292] and performing the contour integration in (2.4), the density of W when $p = 2k + 1$ is odd is given by

$$f(w) = K(p, n)\, p^{(1/2)p(N-p)} w^{(1/2)(N-p)-1}$$
$$\times \left[\sum_{r=k+1}^{\infty} R_r + R + \sum_{l=1}^{k-1} B_l + R_k + \sum_{q=1}^{\infty} G_q + B \right], \qquad (2.5)$$

where R_r, R, B_l, R_k, G_q, and B are given later.

$$R_r = \frac{Ap(W_1^r)[p(r-k)!]}{2^k \Gamma(k)(r-k)! \prod_{i=0}^{k-1} (2r-2i)!} D_{k-1}(W_1, b), \qquad (r > k), \qquad (2.6)$$

where

$$A = \pi^{k/2} 2^{-k(k-2)}, \qquad W_1 = 2^{2k} w / p^p,$$

$$D_{k-1}(W_1, b) = \begin{vmatrix} b_1 & -1 & 0 & 0 & 0 & & 0 \\ b_2 & b_1 & -1 & 0 & 0 & \cdots & 0 \\ b_3 & 2b_2 & b_1 & -1 & 0 & \cdots & 0 \\ b_4 & 3b_3 & 3b_2 & b_1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{k-1} & \binom{k-2}{1} b_{k-2} & \binom{k-2}{2} b_{k-3} & \cdots & \cdots & \cdots & b_1 \end{vmatrix} \qquad (2.7)$$

and b_q's are as in (2.8):

$$b_0 = \log \left[p(r-k)! \Big/ \left\{ (r-k)! \prod_{i=0}^{k-1} (2r-2i)! \right\} \right],$$

$$b_1 = (1 + 2k - p)\, \psi(1) + p \sum_{j=1}^{p(r-k)} (1/j) - \sum_{j=1}^{r-k} (1/j)$$
$$- 2 \sum_{i=0}^{k-1} \sum_{j=1}^{2r-2i} (1/j) - \log W_1, \qquad (2.8)$$

and

$$b_q = (1 + k2^q - p^q)\, \psi_{q-1}(1) + \Gamma(q)$$
$$\times \left[\sum_{j=1}^{p(r-k)} (p/j)^q - \sum_{i=0}^{k-1} \sum_{j=1}^{2r-2i} (2/j)^q - \sum_{j=1}^{r-k} (1/j)^q \right], \qquad q = 2, 3, \ldots .$$

$$R = A\Gamma(k) \prod_{i=1}^{k-1} \Gamma(2i) / \{2\Gamma(kp)\}. \qquad (2.9)$$

$$B_l = \frac{A(W_1)^l \Gamma(k-l) \prod_{i=l+1}^{k-1} \Gamma(2i-2l)}{2^{l+1} \Gamma(l+1)\, \Gamma\{p(k-l)\} [\prod_{i=0}^{l-1} (2l-2i)!]} D_l(W_1, c), \; l = 1, 2, \ldots, k-1, \qquad (2.10)$$

where $D_l(W_1, c)$ is equal to the determinant on the right side of (2.7) with b_1 replaced by c_1, $k-1$ by l and b_q's by c_q's, $q = 2, 3,..., l$. The coefficients c_q's are given by

$$c_1 = \psi(k-l) + 2\left[(l+1)\psi(1) + \sum_{i=l+1}^{k-1} \psi(2i-2l) - \sum_{i=0}^{l-1}\sum_{j=1}^{2l-2i} (1/j)\right]$$
$$- p\psi(pk-pl) - \log W_1,$$

and

$$c_q = \psi_{q-1}(k-l) + 2^q\left[(l+1)\psi_{q-1}(1) + \sum_{i=l+1}^{k-1} \psi_{q-1}(2i-2l) - \sum_{i=0}^{l-1}\sum_{j=1}^{2l-2i} (\Gamma(q)/j^q)\right]$$
$$- p^q\psi_{q-1}(pk-pl), \qquad q = 2, 3,...,$$

where $\sum_{i=n}^{m} (\cdot)$ is interpreted as zero if $n > m$,

$$R_k = \frac{Ap(W_1/2)^k}{\Gamma(k)\prod_{i=0}^{k-1}(2k-2i)!} D_{k-1}(W_1, d), \tag{2.11}$$

where $D_{k-1}(W_1, d)$ can be obtained from (2.7) by replacing b_q's by d_q's, where d_q's are given by

$$d_1 = (1+2k-p)\psi(1) - \log W_1 - \sum_{i=0}^{k-1}\sum_{j=1}^{2k+2i} (2/j)$$

and

$$d_q = (1+2^q k - p^q)\psi_{q-1}(1) - \sum_{i=0}^{k-1}\sum_{j=1}^{2k+2i} (2^q\Gamma(q)/j^q), \qquad q = 2, 3,...,$$

$$G_q = \frac{(-1)^k A W_1^{q+(1/2)}\Gamma(k-\frac{1}{2})\prod_{j=1}^{pq}(j+(p/2)-pk)}{2^k\Gamma(k)\Gamma(kp-(p/2))\prod_{j=1}^{q}(j-k+\frac{1}{2})} D_{k-1}(W_1, f),$$
$$q > 0, \tag{2.12}$$

where $D_{k-1}(W_1, f)$ is the determinant equal to the right side of (2.7) with b_n's replaced by f_n's, where f_n's are given by

$$f_1 = -\log W_1 + \psi(k-\tfrac{1}{2}) + 2k\psi(1) + \sum_{j=1}^{pq}\{p/(j+(p/2)-pk)\}$$
$$- p\psi(kp-(p/2)) - \sum_{l=0}^{k-1}\sum_{j=1}^{2q-2i+1} (2/j),$$

and

$$f_n = \psi_{n-1}(k - \tfrac{1}{2}) + 2k\psi_{n-1}(1) - p^n\psi_{n-1}(pk - (k/2))$$
$$+ \sum_{j=1}^{pq} [p^n\Gamma(n)/(j + (p/2) - pk)^n] - \sum_{i=0}^{k-1} \sum_{j=1}^{2q-2i+1} \left(\frac{2^n\Gamma(n)}{j^n}\right), \quad n = 2, 3,...,$$

and

$$B = -A(W_1)^{1/2}\Gamma(k - \tfrac{1}{2}) \prod_{i=1}^{k-1} (2i - 1)!/\{2\Gamma p(k - \tfrac{1}{2})\}. \tag{2.13}$$

The function $\psi(a) = (d/dx) \log \Gamma(x)|_{x=a}$ and $\psi_j(a) = (d/dx)^j\psi(x)|_{x=a}$. Similarly when $p = 2k$ is even, the density of W is given by

$$f(w) = K(p, n)p^{-(1/2)p(N-p)}w^{(1/2)(N-p)-1}$$
$$\times \left[D + C + \sum_{r=k}^{\infty} D_r + \sum_{l=1}^{k-1} E_l + \sum_{q=k-1}^{\infty} F_q + \sum_{l=1}^{k-2} G_l\right], \tag{2.14}$$

where D, C, D_r, E_l, F_q, and G_l are as given in Nagarsenker and Pillai [11].

(b) *Distribution of W as a Gamma Series*

Using the characteristic function of $\lambda = -2q \log L$, where $L = W^{n/2}$ and q, where $0 < q < \infty$, is an adjustable constant which can be chosen to govern the rate of convergence of the resulting gamma series, the density of λ is obtained as

$$f(\lambda) \equiv K_1(p, n) \sum_{j=0}^{\infty} B_j(2/n)^{j+v}g_{j+v}(2q, x), \tag{2.15}$$

where $K_1(p, n) = K(p, n)(2\pi)^{(p-1)/2}p^{-(pn-1)/2}$, $v = (p^2 + p - 2)/4$,

$$g_\alpha(\beta, x) = [\beta^\alpha\Gamma(\alpha)]^{-1}x^{\alpha-1} e^{-x/\beta},$$

and the coefficients B_j ($j = 1,..., 14$) are as given in Nagarsenker and Pillai [11, Eq. (2.67)]. Formula (2.15) has been derived by using the method of Box [2].

(c) *Distribution of W as a Beta Series*

Using a theorem on the moment function of a random variable proved by Nair [13, p. 175] and Mellin transform, the density of W has been derived as

$$f(w) = K_1(p, n) \sum_{r=0}^{\infty} (B_r)\, w^{(1/2)(N-\lambda)-1}(-\log w)^{v+r-1}/\Gamma(v + r), \tag{2.16}$$

TABLE I

Percentage Points for W from the Exact Distribution and Approximations: Mauchly, Wilks & Tukey, and Box Series

	$p = 3$			
	$N = 8$		$N = 12$	
	5%	1%	5%	1%
Mauchly	0.172	0.083	0.366	0.243
Wilks & Tukey	0.14780	0.07355	0.32344	0.21458
Box series	0.14050	0.06843	0.31842	0.20989
Exact	0.14026	0.06815	0.31836	0.20981
	$p = 5$			
	$N = 7$		$N = 14$	
Wilks & Tukey	0.0^2159	0.0^333	0.11354	0.06852
Box series	0.0^2186	0.0^344	0.11497	0.06957
Exact	0.0^212621	0.0^321839	0.11460	0.069151
	$p = 7$			
	$N = 9$		$N = 15$	
Wilks & Tukey	0.0^345	0.0^311	0.02972	0.01620
Box series	0.0^333	0.0^49	0.02765	0.01491
Exact	0.0^314730	0.0^424239	0.027115	0.014444
	$p = 9$			
	$N = 12$		$N = 16$	
Wilks & Tukey	0.0^322	0.0^47	0.0^2493	0.0^2236
Box series	0.0^325	0.0^48	0.0^2496	0.0^2241
Exact	0.0^313971	0.0^435438	0.0^246163	0.0^221595
	$p = 10$			
	$N = 12$		$N = 16$	
Wilks & Tukey	0.0^428	0.0^57	0.0^2151	0.0^367
Box series	0.0^429	0.0^59	0.0^2141	0.0^363
Exact	0.0^664552	0.0^510036	0.0^212140	0.0^350647

TABLE II

5 % Points of Sphericity Criterion W

N \ p	4	5	6	7	8	9	10
5	$0.0^{4}9528$						
6	$0.0^{2}3866$	$0.0^{4}2578$					
7	0.01687	$0.0^{2}1262$	$0.0^{5}7479$				
8	0.03866	$0.0^{2}6400$	$0.0^{3}4267$	$0.0^{5}2284$			
9	0.06640	0.01650	$0.0^{2}2553$	$0.0^{3}1473$	$0.0^{6}7219$		
10	0.09739	0.03110	$0.0^{2}7004$	$0.0^{3}9434$	$0.0^{4}5149$	$0.0^{6}2326$	
11	0.1297	0.04919	0.01435	$0.0^{2}2950$	$0.0^{3}3631$	$0.0^{4}1817$	$0.0^{7}7722$
12	0.1621	0.06970	0.02433	$0.0^{2}6524$	$0.0^{2}1233$	$0.0^{3}1397$	$0.0^{5}6455$
13	0.1938	0.09174	0.03653	0.01179	$0.0^{2}2924$	$0.0^{3}5114$	$0.0^{4}5370$
14	0.2244	0.1146	0.05051	0.01870	$0.0^{2}5613$	$0.0^{2}1295$	$0.0^{3}2107$
15	0.2535	0.1378	0.06583	0.02712	$0.0^{2}9379$	$0.0^{2}2629$	$0.0^{3}5667$
16	0.2812	0.1608	0.08210	0.03682	0.01423	$0.0^{2}4616$	$0.0^{2}1214$
17	0.3074	0.1835	0.09900	0.04761	0.02011	$0.0^{2}7314$	$0.0^{2}2235$
18	0.3321	0.2058	0.1163	0.05927	0.02693	0.01074	$0.0^{2}3692$
19	0.3533	0.2273	0.1337	0.07161	0.03460	0.01489	$0.0^{2}5630$
20	0.3772	0.2482	0.1511	0.08446	0.04299	0.01973	$0.0^{2}8071$
22	0.4173	0.2876	0.1854	0.1111	0.06154	0.03129	0.01448
24	0.4530	0.3240	0.2185	0.1383	0.08178	0.04494	0.02282
26	0.4848	0.3575	0.2501	0.1654	0.1030	0.06022	0.03287
28	0.5134	0.3882	0.2800	0.1920	0.1248	0.07667	0.04435
30	0.5390	0.4164	0.3081	0.2178	0.1467	0.09392	0.05698
34	0.5833	0.4663	0.3594	0.2665	0.1898	0.1296	0.08468
42	0.6508	0.5453	0.4442	0.3515	0.2697	0.2006	0.1444
50	0.6998	0.6046	0.5106	0.4211	0.3389	0.2660	0.2035
60	0.7447	0.6603	0.5749	0.4910	0.4112	0.3376	0.2715
80	0.8037	0.7354	0.6641	0.5916	0.5196	0.4499	0.3840
100	0.8406	0.7835	0.7228	0.6597	0.5955	0.5317	0.4694
140	0.8842	0.8413	0.7948	0.7453	0.6935	0.6405	0.5870
200	0.9179	0.8868	0.8525	0.8153	0.7757	0.7342	0.6913
300	0.9447	0.9234	0.8996	0.8734	0.8452	0.8151	0.7833

Table continued

TABLE II (*continued*)

1% Points of Sphericity Criterion W

N \ p	4	5	6	7	8	9	10
5	$0.0^{5}3665$						
6	$0.0^{3}6904$	$0.0^{6}9837$					
7	$0.0^{2}5031$	$0.0^{3}2184$	$0.0^{6}2970$				
8	0.01503	$0.0^{2}1828$	$0.0^{4}7187$	$0.0^{7}8604$			
9	0.03046	$0.0^{2}6123$	$0.0^{3}6758$	$0.0^{4}2424$	$0.0^{7}2760$		
10	0.05010	0.01361	$0.0^{2}2498$	$0.0^{3}2520$	$0.0^{5}8306$	$0.0^{8}9216$	
11	0.07258	0.02416	$0.0^{2}6033$	$0.0^{2}1017$	$0.0^{4}9438$	$0.0^{5}2879$	$0.0^{8}3573$
12	0.09679	0.03730	0.01148	$0.0^{2}2646$	$0.0^{3}4120$	$0.0^{4}3544$	$0.0^{5}1004$
13	0.1218	0.05248	0.01880	$0.0^{2}5369$	$0.0^{2}1149$	$0.0^{3}1663$	$0.0^{4}1332$
14	0.1471	0.06915	0.02782	$0.0^{2}9296$	$0.0^{2}2476$	$0.0^{3}4943$	$0.0^{4}6681$
15	0.1721	0.08685	0.03830	0.01444	$0.0^{2}4516$	$0.0^{2}1126$	$0.0^{3}2108$
16	0.1966	0.10518	0.04998	0.02073	$0.0^{2}7343$	$0.0^{2}2160$	$0.0^{3}5065$
17	0.2204	0.1239	0.06261	0.02807	0.01098	$0.0^{2}3669$	$0.0^{2}1018$
18	0.2434	0.1426	0.07595	0.03635	0.01542	$0.0^{2}5707$	$0.0^{2}1804$
19	0.2655	0.1613	0.08982	0.04541	0.02062	$0.0^{2}8300$	$0.0^{2}2914$
20	0.2867	0.1797	0.1040	0.05514	0.02652	0.01146	$0.0^{2}4386$
22	0.3264	0.2156	0.1330	0.07612	0.04017	0.01940	$0.0^{2}8498$
24	0.3626	0.2497	0.1620	0.09845	0.05580	0.02933	0.01421
26	0.3956	0.2819	0.1904	0.1215	0.07287	0.04095	0.02145
28	0.4257	0.3120	0.2180	0.1447	0.09092	0.05392	0.03007
30	0.4531	0.3402	0.2445	0.1677	0.1096	0.06795	0.03989
34	0.5013	0.3910	0.2940	0.2125	0.1475	0.09805	0.06231
42	0.5769	0.4741	0.3789	0.2939	0.2211	0.1611	0.1136
50	0.6331	0.5383	0.4475	0.3632	0.2876	0.2221	0.1671
60	0.6856	0.6001	0.5157	0.4348	0.3594	0.2912	0.2311
80	0.7558	0.6852	0.6129	0.5408	0.4705	0.4034	0.3411
100	0.8006	0.7407	0.6782	0.6144	0.5505	0.4879	0.4275
140	0.8541	0.8085	0.7598	0.7088	0.6562	0.6028	0.5495
200	0.8961	0.8626	0.8262	0.7874	0.7465	0.7040	0.6605
300	0.9297	0.9066	0.8811	0.8535	0.8239	0.7927	0.7600

when v is an integer,

$$= K_1(p, n) \sum_{i=0}^{\infty} R_i w^{(1/2)(N-\lambda)-1}(1 - w)^{v+i-1}/\Gamma(v + i),$$

where the coefficients B_r ($r = 1,..., 14$) and R_i ($i = 1,..., 14$) are given in Nagarsenker and Pillai [11].

3. Computations Using Series Forms and the Approximations

Some of the cases studied are summarized in Tables I and II. Table I gives 5% and 1% points of the exact distribution of W, together with the percentage points as approximated by Mauchly, Box's series, and Wilks–Tukey's approximations for various values of p and N. Table II gives the 5% and 1% points of W for $p = 4(1)10$ and various N. For $\alpha = 0.005, 0.025, 0.1$, and 0.25 and $p = 2(1)10$ percentage points are available in Nagarsenker and Pillai [11].

Table I reveals that even for moderate sample size N, the approximations given by Mauchly [10] for $p = 3$ is extremely poor. Box's series approximation [2] is reasonably good for small values of p and even moderate values of N. Davis' results [5] are generally correct to the decimal he has given but his table is incomplete in regard to small values of N for the values of p he has considered, i.e., $p = 3$, 6, and 10.

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